THE MIXED PROBLEM FOR A MODIFIED MOISTURE-TRANSFER EQUATION

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The motion of moisture in soil is usually described by a nonlinear diffusion equation [1] based on Darcy's law. Yet there have been experiments [2-4] in which the qualitative picture of the moisture content has differed from that given by the solution of the diffusion equation. An attempt to explain the motion of water in accordance with these experimental data was made in [2]. For this purpose the whole network of capillaries in the soil was divided into two groups: wide main channels, through which the bulk of the liquid moves, and fine capillaries, which carry water to the main channels. Denoting the moisture potential in the fine capillaries by ψ and in the main channels by ψ_e (the effective potential in the terminology of [2]) we can write an equation indicating that the difference of these potentials is proportional to the rate of change of moisture content at any depth:

$$\psi_e - \psi = K_1 \partial w / \partial t .$$

Substituting the effective potential ψ_e found in this way in the equation of flow of a single-component compressible fluid [1],

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(a \frac{\partial \psi_e}{\partial x}\right),$$

we obtain a differential equation, which we will call the modified moisture-transfer equation:

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[D(w)\frac{\partial w}{\partial x} + K\frac{\partial^2 w}{\partial t\, \partial x}\right]$$

$$(0 \leqslant x \leqslant H) \qquad \left(D = a\frac{\partial \psi}{\partial w},\ K = aK_1\right). \tag{1}$$

Here D is the diffusivity coefficient and a is the coefficient of moisture conduction. Equation (1) and the arguments leading to it are similar to those which are usually used to describe the motion of a liquid in fissured-porous media [5-8]. The boundary-value problems for such equations are usually formulated for inconsistent initial and boundary conditions.

This paper describes one of the kinds of mixed problems, for which Eq. (1) can be used and gives the solution of this problem in the case of constant coefficients and consistency of the initial and boundary conditions.

We note that although the unidimensional equation (1) is considered here (x is the vertical coordinate) this case is sufficiently characteristic. Conversion to a larger number of measurements introduces only technical difficulties.

To Eq. (1) we add the initial condition

$$w(x, t)\big|_{t=0} = \varphi(x) \tag{2}$$

and the boundary conditions at the ends of the segment

$$\left[D(w)\frac{\partial w}{\partial x} + K\frac{\partial^2 w}{\partial t\, \partial x}\right]\Big|_{x=0} = -f(t), \qquad \frac{\partial w}{\partial x}\Big|_{x=H} = 0 . \tag{3}$$

The physical sense of the second condition of (3) is clear—it is the condition for absence of moisture flow across the boundary x = H. The first condition of (3) is treated as follows. We denote by

$$A(t) = \int_0^H w(x, t)\, dx$$

the moisture content of the layer [0, H] at time t. Integrating Eq. (1) in the limits 0 to H and altering the order of integration and differentiation,

$$\frac{dA}{dt} = \left[D(w)\frac{\partial w}{\partial x} + K\frac{\partial^2 w}{\partial t \partial x}\right]\Big|_{x=0}^{x=H} .$$

Hence, in view of the second condition of (3), we obtain

$$\frac{dA}{dt} = -\left[D(w)\frac{\partial w}{\partial x} + K\frac{\partial^2 w}{\partial t\, \partial x}\right]_{x=0}.$$

The last relationship shows that $f(t) = dA/dt$, i.e., $f(t)$ is the rate of drying of the layer [0, H]. When K = 0 the first condition of (3) becomes a classical condition of the second kind; the physical sense is the same as indicated above. We note that the first boundary relationship (3) was obtained in [6] from other considerations.

We postulate now that $D(\omega)$ = const. This condition is often fulfilled if the range of moisture variation is small. In this case the first condition of (3) is replaced by the simpler condition

$$\partial w / \partial x\, |_{x=0} = f_1(t) \tag{4}$$

$$f_1(t) = -\frac{1}{K}\int_0^t \exp\left\{-\frac{D}{K}(t-\tau)\right\} f(\tau)\, d\tau + \varphi'(0)\exp\left\{-\frac{D}{K}t\right\}. \tag{5}$$

Here $\varphi(x)$ is from Eq. (2). The limiting case K = 0 requires additional consideration, since Eq. (5) is indeterminate for K = 0. We will prove that

$$\lim_{K\to 0} f_1(t) = -\frac{1}{D} f(t) \tag{6}$$

and in this way will show that condition (4) becomes a classical condition of the second kind. For the proof we replace the variable of integration $t - \tau$ by $-K\xi/D$ and we introduce the symbol

$$J(x, t) = \frac{1}{K}\int_0^t \exp\left\{-\frac{D}{K}(t-\tau)\right\} f(\tau)\, d\tau = \frac{1}{D}\int_{-\theta}^0 \exp(\xi)\, f\left(\frac{K}{D}\xi + t\right) d\xi, \quad \theta = \frac{D}{K}t$$

Proceeding from the obvious equality

$$f(t) = \int_{-\infty}^0 \exp(\xi) f(t)\, d\xi$$

we obtain

$$\left|J(x, t) - \frac{f(t)}{D}\right| = \left|\int_{-\theta}^0 \frac{1}{D} e^{\xi} f\left(\frac{K}{D}\xi + t\right) d\xi - \frac{1}{D}\int_{-\infty}^0 e^{\xi} f(t)\, d\xi\right| \leqslant$$

$$\leqslant \left|\int_{-\theta}^0 \frac{1}{D} e^{\xi}\left[f\left(\frac{K}{D}\xi + t\right) - f(t)\right] d\xi\right| + \left|\int_{-\infty}^{-\theta} e^{\xi}\frac{f(t)}{D}\, d\xi\right| .$$

It is easy to see that

$$\lim_{K\to 0}\left|\int_{-\infty}^{-\theta} e^{\xi}\frac{f(t)}{D}\, d\xi\right| = 0.$$

Hence, for an assigned $\varepsilon > 0$ and small K we obtain

$$\left|J(x, t) - \frac{f(t)}{D}\right| \leqslant \int_{-\theta}^0 \left|\frac{1}{D} e^{\xi}\left[f\left(\frac{K}{D}\xi + t\right) - f(t)\right]\right| d\xi +$$

$$+\frac{\varepsilon}{2}\leqslant\frac{1}{D^2}\int_{-\vartheta}^{0}e^{\xi}|f'(t^*)|K|\xi|d\xi+\frac{\varepsilon}{2}=$$

$$=\frac{1}{D^2}|f'(t^*)|\left[K-K\exp\left(-\frac{D}{K}t\right)-\right.$$

$$\left.-Dt\exp\left(-\frac{D}{K}t\right)\right]+\frac{\varepsilon}{2}<\varepsilon,\quad t^*\in[0,t].$$

Here $f'(t)$ is assumed to be bounded. The last inequality results from the possible reduction of K. Since

$$\lim_{K\to 0}\varphi'(0)\exp\left(-\frac{D}{K}t\right)=0,$$

equality (6) is proved.

We proceed to the solution of Eq. (1) with initial condition (2), boundary conditions (4), and the second condition of (3). We assume that the continuity conditions of the limiting relationships $f_1(0) = \varphi'(0)$ and $\varphi'(H) = 0$ are fulfilled. It is easy to find a replacement for the unknown function which will make conditions (2), (4) and the second condition of (3) homogeneous. For instance, $u(x, t) = wf_1(0) - f_1(t)\varphi(x)$ for $f_1(0) = \varphi'(0) \neq 0$ or $u(x, t) = w - \varphi(x)$ for $f_1(0) = \varphi'(0) = 0$. For function u(x, t) the problem then assumes the form

$$\frac{\partial u}{\partial t}=D\frac{\partial^2 u}{\partial x^2}+K\frac{\partial^3 u}{\partial t\,\partial x^2}+F(x,t),$$

$$u|_{t=0}=\frac{\partial u}{\partial x}\Big|_{x=0}=\frac{\partial u}{\partial x}\Big|_{x=H}=0. \tag{7}$$

Here

$$F=-f_1'(t)\varphi(x)+Df_1(t)\varphi''(x)+$$

$$+Kf_1'(t)\varphi''(x)\text{ when }f_1(0)\neq 0$$

and

$$F=D\varphi''(x)\quad\text{when}\quad f_1(0)=\varphi'(0)=0.$$

The solution of the homogeneous equation from (7) is sought in the form X(x)T(t). To determine X we obtain $DX'' = -\lambda_n(X - KX'')$ or

$$X''+\mu_n X=0,\qquad X'(0)=X'(H)=0,\ \mu_n=\lambda_n/(D-K\lambda_n).$$

It is known that nontrivial solutions of X exist only for $\mu_n = n\pi/H$ and are given by $X = B_n \cos\mu_n x$. We will then seek u(x, t) in the form of a series:

$$u(x,t)=\sum_{n=0}^{\infty}u_n(t)\cos\frac{n\pi}{H}x. \tag{8}$$

We expand F(x, t) in a Fourier series:

$$F(x,t)=\sum_{n=0}^{\infty}F_n(t)\cos\frac{n\pi}{H}x,\quad F_n(t)=\frac{2}{H}\int_0^H F(\xi,t)\cos\frac{n\pi}{H}\xi\,d\xi.$$

Substituting expression F(x,t) into (7) we solve the obtained equation

$$u_n'(t)(1+K\mu_n^2)+D\mu_n^2u_n(t)=F_n(t)$$

provided that $u_n(0) = 0$, which follows from (7) and representation (8). We have

$$u_n(t)=\int_0^t\frac{1}{1+K\mu_n^2}\exp\left\{-\frac{D\mu_n^2}{1+K\mu_n^2}(t-\tau)\right\}F_n(\tau)\,d\tau.$$

Substituting this expression and the value of $F_n(t)$ into series (8), we obtain

$$u(x,t)=\int_0^t\int_0^H G(x,\xi,t-\tau)F(\xi,\tau)\,d\xi\,d\tau, \tag{9}$$

$$G(x,\xi,t-\tau)=\frac{2}{H}\sum_{n=0}^{\infty}\frac{1}{1+K\mu_n^2}\times$$

$$\times\exp\left\{-\frac{D\mu_n^2}{1+K\mu_n^2}(t-\tau)\right\}\cos\mu_n x\cos\mu_n\xi. \tag{10}$$

In [8] the solution of another boundary-value problem for Eq. (1) in the case of inconsistent conditions (distribution at the initial instant is linear; at the ends of the segment—zero flow and constant value of the required function, respectively) was given in the form of a similar trigonometric series. The question of the regularity of the presented solution is not considered.

Solution (9) is a series which is majorized by a convergent numerical series

$$M\sum_{n=0}^{\infty}(1+K\mu_n^2)^{-1}$$

if F(x, t) is bounded. Hence, series (9) converges uniformly and, hence, is a generalized solution of problem (7). Representation (10) for the kernel $G(x, \xi, t - \tau)$ shows directly that the series for $\partial u/\partial t$ converges uniformly. Let the limit functions $f(t)$ and $\varphi(x)$ be such that almost everywhere on [0, H] there is a bounded derivative $\partial^2F(x, t)/\partial x^2$. In this case the sequence of equalities obtained by successive integration by parts and the use of the equalities

$$\frac{\partial G}{\partial\xi}\Big|_{\xi=0}=\frac{\partial G}{\partial\xi}\Big|_{\xi=H}=0$$

show the existence of a second derivative of the function u(x, t):

$$\frac{\partial^2 u}{\partial x^2}=\int_0^t\int_0^H\frac{\partial^2 G}{\partial x^2}F(\xi,\tau)\,d\xi\,d\tau=\int_0^t\int_0^H\frac{\partial^2 G}{\partial\xi^2}F(\xi,\tau)\,d\xi\,d\tau=$$

$$=-\int_0^t G\frac{\partial F}{\partial\xi}\,d\tau\Big|_0^H+\int_0^t\int_0^H G\frac{\partial^2 F}{\partial\xi^2}\,d\xi\,d\tau.$$

In a similar way we can show the existence of the derivative $\partial^3u/\partial t\partial x^2$. Thus, if the additional continuity conditions for the functions $f(t)$ and $\varphi(x)$ are fulfilled, formula (9) gives a regular solution of problem (7).

For the operator L and its conjugate operator M, according to Lagrange,

$$Lu=\frac{\partial u}{\partial t}-D\frac{\partial^2 u}{\partial x^2}-K\frac{\partial^3 u}{\partial t\,\partial x^2},\quad Mu=-\frac{\partial u}{\partial t}-D\frac{\partial^2 u}{\partial x^2}+K\frac{\partial^3 u}{\partial t\,\partial x^2},$$

we have a formula which is easily verified:

$$u(x,t)Mv(x,t)-v(x,t)Lu(x,t)=$$

$$=-\frac{\partial}{\partial t}\left(uv+K\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right)+$$

$$+\frac{\partial}{\partial x}\left[D\left(v\frac{\partial u}{\partial x}-u\frac{\partial v}{\partial x}\right)+K\left(v\frac{\partial^2 u}{\partial t\partial x}+u\frac{\partial^2 v}{\partial t\partial x}\right)\right]. \tag{11}$$

Function $G(x, \xi, t - \tau)$ is a generalized solution of the equation $L_{x,t}G = 0$ (the subscripts to the operator L denote that the derivatives are taken with respect to these variables). It is easy to confirm that

$$\int_0^t\int_0^H M_{\xi,\tau}G(x,\xi,t-\tau)\,d\xi\,d\tau=0$$

i.e., that G as a function of the arguments ξ and τ satisfies the equation $MG = 0$ in the generalized sense. In (11) we put $u = f_1(\tau)\varphi(\xi)$, $v = G(x, \xi, t - \tau)$ and integrate the equality in the limits [0, H] and [0, t] with respect to ξ and τ, respectively:

$$-\int_0^t\int_0^H G(x,\xi,t-\tau)L[f_1(\tau)\varphi(\xi)]\,d\xi\,d\tau=$$

$$= \int_0^H [f_1(0)\,G(x,\xi,t) - f_1(t)\,G(x,\xi,0)]\,\varphi(\xi) +$$
$$+ \int_0^t D\Big[G(x,0,t-\tau)\,f_1(\tau)\,\varphi'(0) -$$
$$- f_1(\tau)\,\varphi(0)\,\frac{\partial G(x,0,t-\tau)}{\partial \xi}\Big]\,d\tau +$$
$$+ \int_0^t K\,[G(x,0,t-\tau)\,f_1'(\tau)\,\varphi'(0) +$$
$$+ f_1(\tau)\,\varphi(0)\,\frac{\partial^2 G(x,0,t-\tau)}{\partial\tau\,\partial\xi}\Big]\,d\tau - \int_0^t f_1(\tau)\,\varphi(H) \times$$
$$\times \Big[K\frac{\partial^2 G(x,H,t-\tau)}{\partial\tau\,\partial\xi} - D\frac{\partial G(x,H,t-\tau)}{\partial\xi}\Big]\,d\tau +$$
$$+ K\int_0^H \Big[f_1(0)\,\frac{\partial G(x,\xi,t)}{\partial\xi} - f_1(t)\,\frac{\partial G(x,\xi,0)}{\partial\xi}\Big]\,\varphi'(\xi)\,d\xi. \quad (12)$$

We assume that $f_1(t)$, $\varphi(x)$, and their derivatives are bounded. Then, performing integration by parts on the right side of equality (12) and transferring the differentiation to $\varphi(\xi)$ and $f_1(\tau)$ we note that integrals with kernels $\partial G/\partial\xi$ and $\partial^2 G/\partial\tau\partial\xi$ converge uniformly, since the series for G is majorized by a convergent numerical series. We will assume further that $L_{\xi,\tau}[f_1\varphi] = F(\xi,\tau)$, $\varphi'(0) = f_1(0)$ and $\varphi'(H) = 0$. In addition, representation (10) directly shows that

$$\frac{\partial G(x,0,t-\tau)}{\partial\xi} = \frac{\partial G'(x,H,t-\tau)}{\partial\xi} = 0.$$

Taking into account what has been said, and also equality (9), we obtain from formula (12)

$$u(x,t) = \int_0^H f_1(t-\tau)\Big[\varphi G(x,\xi,\tau) + K\varphi'\frac{\partial G(x,\xi,\tau)}{\partial\xi}\Big]\Big|_{\tau=0}^{\tau=t} d\xi +$$
$$+ \varphi'(0)\int_0^t [Df_1(\tau) + Kf_1'(\tau)]\,G(x,0,t-\tau)\,d\tau. \quad (13)$$

In a similar way we can use (11) to express u(x, t) in the case in which $f_1(0) = \varphi'(0) = 0$:

$$u(x,t) = -f_1(t)\int_0^H \Big[\varphi G(x,\xi,0) + K\varphi'\frac{\partial G(x,\xi,0)}{\partial\xi}\Big]\,d\xi =$$
$$= \frac{1}{K}\Big[\int_0^t \exp\Big\{-\frac{D}{K}(t-\tau)\Big\}\,f(\tau)\,d\tau\Big]\int_0^H \Big[\varphi G(x,\xi,0) +$$
$$+ K\varphi'\frac{\partial G(x,\xi,0)}{\partial\xi}\Big]\,d\xi.$$

We note that as $K \to 0$, function $G(x, \xi, t-\tau)$ has a limit $G_0(x, \xi, t-\tau)$, which is the Green's function of the problem

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \quad u|_{t=0} = \varphi(x),$$
$$D\frac{\partial u}{\partial x}\Big|_{x=0} = -f(t), \quad \frac{\partial u}{\partial x}\Big|_{x=H} = 0.$$

Formula (9) in this case after conversion from u(x, t) to w(x, t) gives the solution of this problem.

The disagreement between the experimental data and the solution of the diffusion equation, which was noted at the beginning of the paper, is as follows. If at the initial instant the upper layer of soil is moister than the lower and intense evaporation takes place, then, according to the experimental data [2-4], the moisture content of the lower, drier layers decreases as the upper layer dries out. If the motion of the water obeys the diffusion equation, the gradient of the moisture content between the upper and lower layers will create a flow of moisture into the lower layer, at least at the start of the process, i.e., a flow against the moisture gradient.

Using formula (13) we can illustrate how Eq. (1) simulates the motion of moisture along the moisture gradient. We assume for simplicity that $f(t)$ is chosen so that $f_1(t) = f_1(0) =$ const. The expression for the moisture content will then take the form $w = a^{-1}u(x,t) + \varphi(x)$. When $K = 0$ (13) becomes the usual solution of the diffusion problem $u_0(x,t)$ and, hence, to demonstrate the motion of moisture along the moisture gradient it is sufficient to show that when $K < 0$ the inequality $u_0(x,t)\,u(x,t) < 0$ can be satisfied.

If we assume t is so small that the second term of the right side of (13) can be neglected and K is such that

$$\int_0^H \Big[\varphi G(x,\xi,0) + K\varphi'\frac{\partial G(x,\xi,0)}{\partial\xi}\Big]\,d\xi < 0,$$

the inequality can be satisfied if a decreasing function $\varphi(x) > 0$, $\varphi'(x) < 0$ is chosen as the initial distribution. We note that the choice of a decreasing function as φ is natural if we wish to explain the "anomalous" movement along the moisture gradient. In fact, if $\varphi' \equiv 0$, the reduction of moisture content at x = H is a consequence of evaporation. In the case of $\varphi' > 0$ diffusion also promotes a reduction of moisture content at x = H.

In [2] the case of a diminishing initial moisture distribution was experimentally verified. An increase in K can lead not only to a reduction in the influx of moisture at the bottom of the layer (x = H), but also to an outflow of moisture, i.e., the experimentally observed effect.

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